Note: In this problem set, expressions in green cells match corresponding expressions in the text answers.

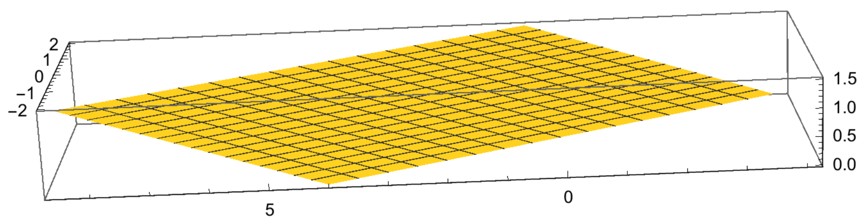
1 - 10 Flux integrals (3) $\int_S \mathbf{F.n}\, dA$

Evaluate the integral for the given data. Describe the kind of surface.

1. $F = \{-x^2, y^2, 0\}$, $S : r = \{u, v, 3u - 2v\}$, $0 \le u \le 1.5$, $-2 \le v \le 2$

```
Clear["Global`*"]
```

```
ParametricPlot3D[{u, v, 3 u - 2 v}, {u, 0, 1.5}, {v, -2, 2}]
```



This is a plane. The parametric expression for r is already available.

```
gotoe[u_, v_] = {u, v, 3 u - 2 v}
{u, v, 3 u - 2 v}
```

Taking the partial derivatives to get ready for calculating the normal vector.

```
fir = D[{u, v, 3 u - 2 v}, {u}]
{1, 0, 3}
sec = D[{u, v, 3 u - 2 v}, {v}]
{0, 1, -2}
```

Then finding the normal vector.

```
norm = Cross[fir, sec]
{-3, 2, 1}
```

At this point the s.m. explains that it is time to substitute the elements of r into F

$F2 = \{-u^2, v^2, 0\}$

Then take the dot product F2 . norm. (Unfortunately *Mathematica* won't accept symbolic arguments here.)

$dotFN = \{-u^2, v^2, 0\}.\{-3, 2, 1\}$

$3u^2 + 2v^2$

$\int\int_S$ (dotFN) dA will be essentially what I will be looking for next.

$$\int_{-2}^{2}\int_{0}^{1.5}\left(3\,u^2 + 2\,v^2\right)\,du\,dv$$

```
29.5
```

The value shown on the above line is the text's answer to the problem.

3. $F = \{0, x, 0\}$, $S: x^2 + y^2 + z^2 = 1$, $x \geq 0$, $y \geq 0$, $z \geq 0$

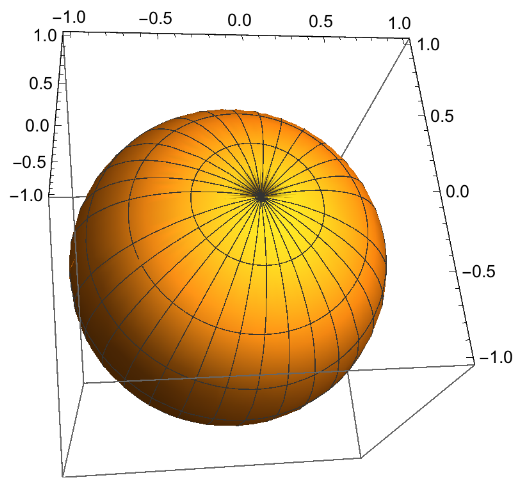
```
Clear["Global`*"]
```

It will be necessary to parameterize S. This will be simpler looking than the sphere in Sec 10.5, because the center is at the origin, and the root of the radius expression is 1.

```
sph[u_, v_] = {Cos[v] Cos[u], Cos[v] Sin[u], Sin[v]}
{Cos[u] Cos[v], Cos[v] Sin[u], Sin[v]}
```

I had a consistent sphere parameterization, but changed it so that it would match the text's version, swapping u and v, essentially.

```
ParametricPlot3D[
 {Cos[v] Cos[u], Cos[v] Sin[u], Sin[v]}, {u, 0, π}, {v, 0, 2 π}]
```



Taking the partial derivatives to get ready for calculating the normal vector.

```
fir = D[{Cos[v] Cos[u], Cos[v] Sin[u], Sin[v]}, {u}]
{-Cos[v] Sin[u], Cos[u] Cos[v], 0}
```

```
sec = D[{Cos[v] Cos[u], Cos[v] Sin[u], Sin[v]}, {v}]
{-Cos[u] Sin[v], -Sin[u] Sin[v], Cos[v]}
```

Then finding the normal vector.

```
norm = Simplify[Cross[fir, sec]]
{Cos[u] Cos[v]^2, Cos[v]^2 Sin[u], Cos[v] Sin[v]}
```

At this point it is time to substitute the elements of r into F

```
F = {0, Cos[v] Cos[u], 0}
{0, Cos[u] Cos[v], 0}
```

Then take the dot product F2 . norm. (This time Mathematica accepts the symbolic reference.)

```
dotp = F.norm
```

$\text{Cos}[u]\ \text{Cos}[v]^3\ \text{Sin}[u]$

The answer on the line above matches the text's. $\int\int_S (\text{dotp})\, dA$ will be essentially what I will be looking for next.

$$\int_0^{\pi/2}\int_0^{\pi/2} (\text{Cos}[u]\ \text{Cos}[v]^3\ \text{Sin}[u])\, du\, dv$$

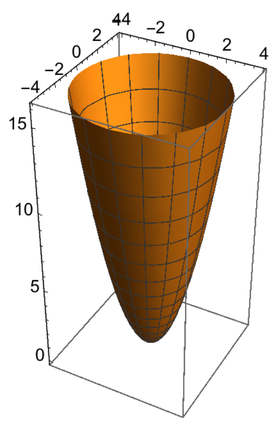
$$\frac{1}{3}$$

The above answer matches the text's. I played with the limits until it came out right. If either integral limit goes to 2pi, or even to pi, the answer goes to zero. However, in the plot, u needs to go to pi, and v to 2pi, in order to draw a complete sphere. The text mentioned something about projecting the surface onto a plane.

5. $F = \{x, y, z\}$, $S : r = \{u \text{Cos}[v], u \text{Sin}[v], u^2\}$, $0 \le u \le 4$, $-\pi \le v \le \pi$

```
Clear["Global`*"]
```

$\text{ParametricPlot3D}[\{u \text{Cos}[v], u \text{Sin}[v], u^2\}, \{u, 0, 4\}, \{v, -\pi, \pi\}]$



It's a paraboloid! Let the function be so defined.

$\text{parab}[u\_, v\_] = \{u \text{Cos}[v], u \text{Sin}[v], u^2\}$

$\{u \text{Cos}[v], u \text{Sin}[v], u^2\}$

And take the partial derivatives

```
fir = D[{u Cos[v], u Sin[v], u^2}, {u}]
{Cos[v], Sin[v], 2 u}

sec = D[{u Cos[v], u Sin[v], u^2}, {v}]
{-u Sin[v], u Cos[v], 0}
```

Then cross them.

```
norm = Cross[fir, sec]
{-2 u^2 Cos[v], -2 u^2 Sin[v], u Cos[v]^2 + u Sin[v]^2}
```

Then express **F** as itself with **r**'s components replacing **F**'s native components.

```
F = {u Cos[v], u Sin[v], u^2}
{u Cos[v], u Sin[v], u^2}
```

Dot the modified **F** with norm, the cross.

```
dotp = Simplify[F.norm]
```

$-u^3$

And integrate.

$$\int_{-\pi}^{\pi}\int_{0}^{4}\left(-u^3\right)\,du\,dv$$

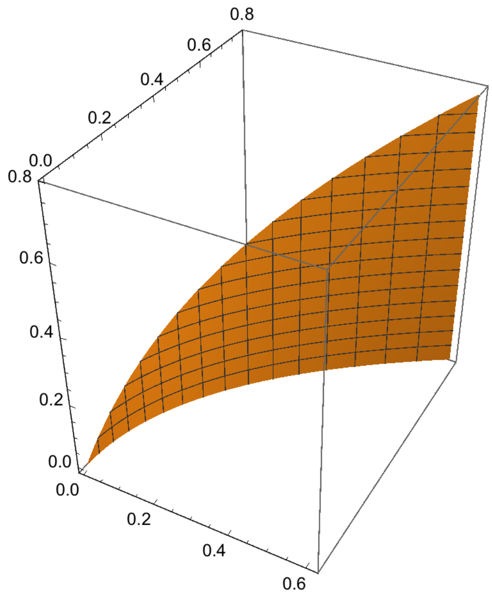
$-128\,\pi$

Both cells in blue match the text answers.

7. $F = \{0, \mathrm{Sin}[y], \mathrm{Cos}[z]\}$,

S the cylinder $x = y^2$, where $0 \le y \le \frac{\pi}{4}$ and $0 \le z \le y$

```
ParametricPlot3D[{u^2, u, v}, {u, 0, π / 4}, {v, 0, u}, ImageSize → 250]
```



I believe this can be called a cylinder, if not a closed cylinder.

```
parcyl[u_, v_] = {u^2, u, v}
{u^2, u, v}

fir = D[{u^2, u, v}, {u}]
{2 u, 1, 0}

sec = D[{u^2, u, v}, {v}]
{0, 0, 1}

norm = Cross[fir, sec]
{1, -2 u, 0}

F = {0, Sin[u], Cos[v]}
{0, Sin[u], Cos[v]}

dotted = F.norm
-2 u Sin[u]
```

Below I flip the order of du and dv so I can put the symbolic limit on the interior. I had trouble with the limit on u. But I finally got it right. The plot shows what it looks like.

```
outt = ∫_0^(π/4) ∫_0^u (-2 u Sin[u]) ⅆv ⅆu
```

$$\frac{1}{16}\left(-32\left(-2+\sqrt{2}\right)+\sqrt{2}\,(-8+\pi)\,\pi\right)$$

```
PossibleZeroQ[outt - (4 + (-2 + π^2/16 - π/2) √2)]
True
```

With the above test, I can put the blue color on the outt cell, showing equality with the text.

```
N[outt]
```

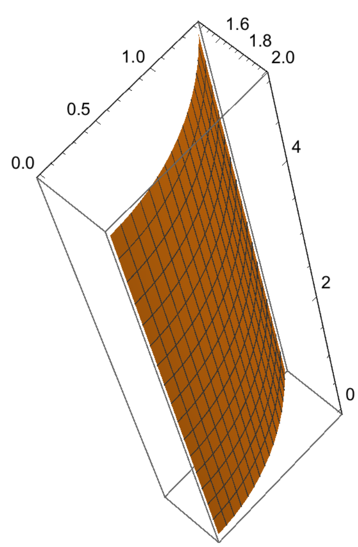
-0.177511

And the numerical version, above, also agrees with the text.

9. F = {0, Sinh[z], Cosh[x]},

S : $x^2 + z^2 = 4$, $0 \le x \le \frac{1}{\sqrt{2}}$, $0 \le y \le 5$, $z \ge 0$

```
Clear["Global`*"]
ParametricPlot3D[{2 Cos[u], 2 Sin[u], v}, {u, 0, π / 4}, {v, 0, 5}]
```



The below parametric expression is the form the text uses.

```
sphf[u_, v_] = {2 Cos[u], 2 Sin[u], v}
{2 Cos[u], 2 Sin[u], v}

fir = D[{2 Cos[u], 2 Sin[u], v}, {u}]
{-2 Sin[u], 2 Cos[u], 0}
```

```
sec = D[{2 Cos[u], 2 Sin[u], v}, {v}]
{0, 0, 1}

norm = Cross[fir, sec]
{2 Cos[u], 2 Sin[u], 0}

F = {0, Sinh[v], Cosh[2 Cos[u]]}
{0, Sinh[v], Cosh[2 Cos[u]]}

dotted = F.norm
2 Sin[u] Sinh[v]
```

Below: u is given the evaluation limits of y, which makes sense. However, I don't see why the limits assigned for v are chosen. These limits are included in the text's answer.

$$\text{outt} = \int_0^5 \int_0^{\pi/4} (2\,\text{Sin}[u]\,\text{Sinh}[v])\, du\, dv$$

$$-\left(-2 + \sqrt{2}\right)(-1 + \text{Cosh}[5])$$

$$\text{PossibleZeroQ}\left[\text{outt} - 2\left(1 - 1\middle/\sqrt{2}\right)(\text{Cosh}[5] - 1)\right]$$

```
True

N[outt]
42.8854
```

12 - 16 Surface integrals (6) $\int_S \int G(\mathbf{r})\, dA$

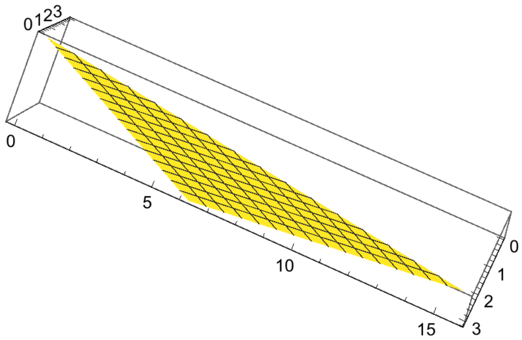
Evaluate these integrals for the following data. Indicate the kind of surface.

13. G = x + y + z, z = x + 2 y, 0 <= x <= π, 0 <= y <= x

```
Clear["Global`*"]
```

I will try to use problem 15, worked by the s.m., as a guide to doing this one.

```
ParametricPlot3D[{u, v, 2 u + 3 v}, {u, 0, π}, {v, 0, u}]
```




```
pln[u_, v_] = {u, v, 2 u + 3 v}
{u, v, 2 u + 3 v}

fir = D[{u, v, 2 u + 3 v}, {u}]
{1, 0, 2}

sec = D[{u, v, 2 u + 3 v}, {v}]
{0, 1, 3}

norm = Cross[fir, sec]
{-2, -3, 1}

nsq = norm.norm
14

Gr = {u + v + 2 u + 3 v}
```

outt = $\int_0^{\pi}\int_0^{u} (u + v + 2u + 3v)\, 14\, dv\, du$

$\frac{70\pi^3}{3}$

```
N[%]
```

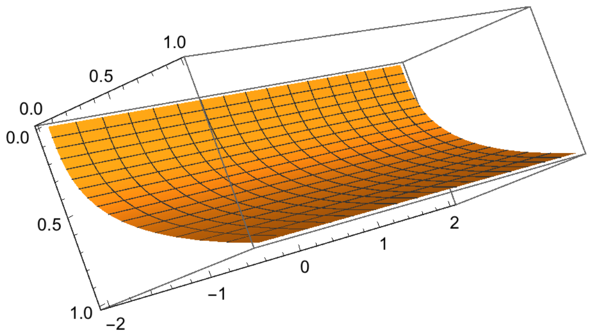
42.8854

The above is probably not too close, though it does have the π^3 term, which may mean the correct limit for u was used. There are no hints in the answer to help me try to get closer. Text answer = $\frac{7\pi^3}{\sqrt{6}}$=88.6.

15. G = $(1 + 9xz)^{\frac{3}{2}}$, S : r = $\{u, v, u^3\}$, $0 \le u \le 1$, $-2 \le v \le 2$

```
Clear["Global`*"]
```

```
ParametricPlot3D[{u, v, u^3}, {u, 0, 1}, {v, -2, 2}]
```




```
cyl[u_, v_] = {u, v, u^3}
```

$\{u, v, u^3\}$

```
fir = D[{u, v, u^3}, {u}]
```

$\{1, 0, 3 u^2\}$

```
sec = D[{u, v, u^3}, {v}]
```

{0, 1, 0}

```
norm = Cross[fir, sec]
```

$\{-3 u^2, 0, 1\}$

Since this problem deals with a surface without orientation, a |norm| factor needs to be included in the integration. This hint comes from the s.m..

```
nsq = norm.norm
```

$1 + 9 u^4$

So that |norm| is the square root of the above

```
sqru = Sqrt[nsq]
```

$\sqrt{1 + 9 u^4}$

Next is the part about using the cartesian form to host the parametric form. In the parametric form the x position is held by u, and z position is held by u^3. Therefore

$Gr = (1 + 9 u u^3)^{3/2}$

$(1 + 9 u^4)^{3/2}$

There is an additional term in the integral, corresponding to the |N| term, above called sqru.

outt = $\int_{-2}^{2}\int_{0}^{1}\left(\sqrt{1+9u^4}\,\left(1+9u^4\right)^{3/2}\right)du\,dv$

$\frac{272}{5}$

$\frac{272.}{5}$

54.4

The above line agrees with the text’s answer.

21. Find a formula for the moment of inertia of the lamina in problem 20 about the line y = x, z = 0.

22 - 23 Find the moment of inertia of a lamina S of density 1 about and axis B, where

23. S : $x^2 + y^2 = z^2$, $0 \le z \le h$, B : the z - axis

25. Using Steiner’s theorem, find the moment of inertia of a mass of density 1 on the sphere S: $x^2+y^2+z^2 = 1$ about the line K: x = 1, y = 0 from the moment of inertia of the mass about a suitable line B, which you must first calculate.